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Stable Kernel Representations and the Youla Parameterization for Nonlinear Systems

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Abstract

In this paper a general approach is taken to yield a characterization of the class of stable plant controller pairs, which is a generalization of the Youla parameterization for linear systems. This is based on the idea of representing the input-output pairs of the plant and controller as elements of the kernel of a related operator, denoted the kernel representation of the system. Results giving one method of deriving a kernel representation for a nonlinear plant with a general state space description are presented.

1. Introduction

A method of representing nonlinear systems is presented which we denote the *kernel representation* of the system. The input-output pairs of a system may be found in the kernel of this related operator, which maps from the combined input and output spaces to some other space. This has obvious links to the Behavioral approach to control developed by Willems, see for example [17, 18], and the references therein. In this paper we do not explore these links, we develop a framework in which kernel representations may be used in the definition of such concepts as well-posedness and stability of a closed loop system, and investigate their role as a generalization of left coprime factorizations of nonlinear operators.

It is demonstrated that with the formalism derived, the class of plants stabilized by a given controller, the class of controllers stabilizing a given plant, and the class of all stable plant controller pairs may be easily parameterized. This mimics the results of [7], where such results were obtained using left coprime factorizations of the plant and controller, and the linear results of [14], which uses the Youla parameterization. The results presented in this paper are, however, more general. Firstly as they are applicable to a wider class of systems, and secondly as they are derived without distinguishing between the input and output spaces of the plant and controller. The development of the relationship between the kernel representation and the input-output representation of the system is delayed until after the presentation of the main results in order to emphasize the latter fact.

This work continues a series of investigations into the use of coprime factorizations in nonlinear systems analysis. Specifically, the motivation for these results is due to the use of left coprime factorizations of nonlinear systems,

where a version of the Youla parameterization, giving the class of all linear controllers stabilizing a linear plant, has been derived, see [7] or [6] for details. This line of investigation was initiated by Hammer [1, 2, 3], and further developed by Tay [13] and Paice and Moore [6, 8].

The main weakness of the nonlinear factorization theory is the lack of results linking the state-space and input-output theories. By considering a right factorization approach, see e.g. Verma [15] or Sontag [12], right factorizations may be derived from the state-space description of a nonlinear system. However, analogues of the Youla parameterization are not derivable within this framework. The Youla parameterization has been derived based on left coprime factorizations, but to date state space formulations for nonlinear left coprime factorizations have only been derived for special cases, see e.g. [4].

The framework presented here is the first such framework that yields a Youla parameterization for nonlinear systems and naturally admits state space descriptions of all input-output results obtained.

The paper is organized as follows. In Section 2 a general framework for using stable kernel representations is presented. The concepts of well-posedness and stability of feedback systems are developed for use within this framework. The main results of the paper are presented in Section 3, giving the class of stable closed loop systems which are representable within this framework. This class is parameterized in a way which specializes to the Youla parameterization in the linear case. The relationship of the *skr* of a system to its input-output representation is then developed yielding more direct versions of the Youla parameterization are derived. In Section 4 a state-space approach to deriving stable kernel representations due to Scherpen and van der Schaft [11] is presented. Conclusions are drawn in Section 5.

2. Kernel Representations

In the sequel the term system will be taken to denote a general (dynamical) system, and the terms feedback system or closed-loop system will be used to indicate an interconnection of such systems.

Representing a General System

Consider the system Σ , with input and output spaces \mathcal{U} and \mathcal{Y} respectively, and initial condition space \mathcal{X}_Σ . Note

that \mathcal{U} and \mathcal{Y} are taken to be signal spaces, that is vector spaces of functions from a given time domain to a given Euclidean space, whereas the initial condition space \mathcal{X}_Σ is more commonly a Euclidean vector space. It is assumed that every such system under consideration may be described by a family of maps

$$R_\Sigma^x : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}, \quad \forall x \in \mathcal{X}_\Sigma, \quad (1)$$

known as the *kernel representation* of Σ , such that all possible input-output pairs u, y for the system Σ with initial conditions $x \in \mathcal{X}_\Sigma$ satisfy:

$$R_\Sigma^x(y, u) = 0. \quad (2)$$

Remark 2.1 For the subsequent developments, until Section 3.2, it is in fact not necessary to distinguish *a priori* between inputs and outputs. Indeed, if we group u and y into a single vector w , then the entire framework and all results will remain valid for systems described as $R_\Sigma^x(w) = 0$, where R_Σ^x is an operator from \mathcal{W} (the space of external signals) to \mathcal{Z} . This is clearly related to the behavioral approach to control, see e.g. [17, 18].

In general it is not possible to describe a kernel representation by a single map $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}$, however for brevity, we shall refer to the kernel representation R_Σ . The key to the development of the following results is to examine the solutions to

$$R_\Sigma^x(y, u) = z \quad (3)$$

where z is not necessarily equal to zero.

For z arbitrary, the input-output map induced by the solution pairs to (3) for a given initial condition $x \in \mathcal{X}$ will be denoted by $\Sigma_x(x) : \mathcal{U} \mapsto \mathcal{Y}$, although as noted above, the following development is not dependent on the existence of this map. The input-output map $\Sigma_0(x) : \mathcal{U} \mapsto \mathcal{Y}$ will be simply denoted by $\Sigma(x)$; the input-output map of Σ for initial condition x .

Remark 2.2 Note that every system has a kernel representation, although it will not be unique, for example any input-output map $\Sigma(x) : \mathcal{U} \mapsto \mathcal{Y}$ has kernel representation $R_\Sigma^x(y, u) = F(y - \Sigma(x)u)$, for any invertible $F : \mathcal{Y} \mapsto \mathcal{Z}$ such that $F(0) = 0$.

Feedback Systems

In this subsection the notion of interconnecting two systems, the plant and the controller, to form closed-loop or feedback systems is introduced and developed for use within this framework. Note that it is common to allow for the introduction of external signals between the plant and controller so as to account for reference signals, or noise signals corrupting the control or measured signal, see for instance [7]. In the sequel only the case where these external signals are zero will be considered. This is referred to as the noise-free case.

Consider a plant, $G : \mathcal{U} \rightarrow \mathcal{Y}$, and controller $K : \mathcal{Y} \rightarrow \mathcal{U}$, with kernel representations $R_G : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_G$, $R_K : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}_K$ respectively, which are interconnected to form the system $\{G, K\}$ as in Figure 1. The closed loop then has a kernel representation

$$R_{\{G, K\}} : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_G \times \mathcal{Z}_K \quad (4)$$

$$R_{\{G, K\}} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} R_G(y, u) \\ R_K(u, y) \end{pmatrix}$$

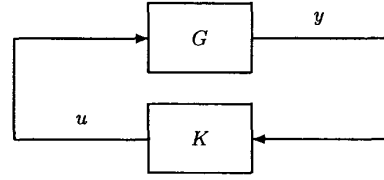


Figure 1: The system $\{G, K\}$, without external inputs

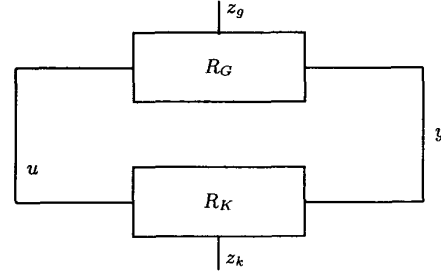


Figure 2: The Kernel Representation of $\{G, K\}$.

as in Figure 2.

The existence of a solution pair (u, y) for a given (z_G, z_K) is not guaranteed. Thus, in order to work with feedback systems within this framework, we will need to assume that solutions exist. This property is known as well-posedness.

Definition 2.3 The system $\{G, K\}$ is well-posed iff for all initial conditions, $(x_G, x_K) \in \mathcal{X}_G \times \mathcal{X}_K$, and for all $(z_G, z_K) \in \mathcal{Z}_G \times \mathcal{Z}_K$, the solution (u, y) to (4) is unique. That is, for all $(x_G, x_K) \in \mathcal{X}_G \times \mathcal{X}_K$,

$$\left[R_{\{G, K\}}^{z_G, z_K} \right]^{-1} : \mathcal{Z}_G \times \mathcal{Z}_K \rightarrow \mathcal{U} \times \mathcal{Y} \quad \text{exists.} \quad (5)$$

□

Remark 2.4 The above definition of well-posedness of a feedback system, when specialized to linear systems, is very similar to the notion of *regular feedback interconnection*, as proposed in [19]. Note that the requirement of existence of unique solutions for every $x \in \mathcal{X}$ excludes the possibility of *singular* feedback ([19]).

In the sequel, the well-posedness of such feedback systems will be considered over cross products of signal and initial condition spaces. This will be indicated in the subscripts, e.g. $\mathcal{Z}_{GK} = \mathcal{Z}_G \times \mathcal{Z}_K$, $z_{GK} = (z_G, z_K) \in \mathcal{Z}_{GK}$.

Stability

We now define the concept of stability for general nonlinear operators and feedback systems. This is defined implicitly via the notion of stability on the various input and output spaces of these operators. A signal space \mathcal{Z} is divided into two disjoint subsets as follows

$$\mathcal{Z} = \mathcal{Z}^s \cup \mathcal{Z}^u, \quad \mathcal{Z}^s \cap \mathcal{Z}^u = \emptyset,$$

where \mathcal{Z}^s denotes the set of all stable signals, and \mathcal{Z}^u the set of all unstable signals. For the space \mathcal{Z}_{GK} ; \mathcal{Z}_{GK}^s is

defined to be $Z_G^s \times Z_K^s$, and Z_{GK}^u is the remainder of the space.

Note that Z may be partitioned in many ways. It is not assumed that Z^s is a vector space, or that it is closed, although it is assumed that $0 \in Z^s$. Commonly these sets are formed by defining a norm on the space Z , and then defining a signal to be stable iff it has finite norm.

Definition 2.5 An input-output map $\Sigma : \mathcal{U} \rightarrow \mathcal{Y}$ is said to be stable if the image of \mathcal{U}^s under Σ is a subset of \mathcal{Y}^s . \square

Definition 2.6 A kernel representation $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow Z$ of Σ is called a stable kernel representation (skr) of Σ iff for all initial conditions $x \in \mathcal{X}_\Sigma$, $R_\Sigma^x(\cdot, \cdot)$ is a stable operator. That is, if $y \in \mathcal{Y}^s$, $u \in \mathcal{U}^s$, then $z = R_\Sigma^x(y, u) \in Z^s$. \square

Unless otherwise stated, all kernel representations used in the sequel will be skrs.

The definition of stability is now extended to include closed loop systems.

Definition 2.7 The closed loop system $\{G, K\}$ with skr $R_{\{G,K\}}$ as in (4) is stable over $\mathcal{B}_{GK} \subset Z_{GK}^s \times \mathcal{X}_{GK}$ if it is well-posed, and for all pairs $(z_{GK}, x_{GK}) \in Z_{GK}^s \times \mathcal{X}_{GK}$ the solution (y, u) to (4) is stable, iff $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$.

A section of \mathcal{B}_{GK} corresponding to the initial condition x is denoted $\mathcal{B}_{GK}^x = \{z : (z, x) \in \mathcal{B}_{GK}\}$. \square

The system $\{G, K\}$ is said to be *generally stable*, or simply *stable*, if it is stable over $Z_{GK}^s \times \mathcal{X}_{GK}$. The system signals (u, y) must be unstable for $z_{GK} \in Z_{GK}^u$, otherwise the stability of the kernel representations R_G and R_K would be contradicted.

Lemma 2.8 The system $\{G, K\}$ is well-posed and stable over $\mathcal{B} \subset Z_{GK}^s \times \mathcal{X}_{GK}$ iff for all $x_{GK} \in \mathcal{X}_{GK}$ the map

$$\left[R_{\{G,K\}}^{x_{GK}} \right]^{-1} : Z \rightarrow \mathcal{Y} \times \mathcal{U} \text{ exists,} \quad (6)$$

$$\text{and } \left[R_{\{G,K\}}^{x_{GK}} \right]^{-1} \mathcal{B}^{x_{GK}} \subset \mathcal{Y}^s \times \mathcal{U}^s \quad (7)$$

\square

The proof arises out of the definitions and is left to the reader.

3. Main Results

In this section the results of [6, 8] giving nonlinear versions of the Youla parameterization are generalized to use the framework presented in the previous section.

The construction of a general well-posed and stable class of plant-controller pairs from a given well-posed and generally stable feedback system $\{G, K\}$ with skr (4) is first presented. It is shown that this generates the class of all well-posed and stable feedback systems which are expressible within this framework. A specialization of our framework which admits a well-defined input-output operator for each kernel representation is then developed. The Youla parameterizations of the class of stabilizing controllers for a given plant, and the class of plants stabilized by a given controller are then stated.

Due to limitations of space, the results presented in this paper have been specialized to consider only the case that the system $\{G, K\}$ is generally stable. The results are presented in their full generality in [9].

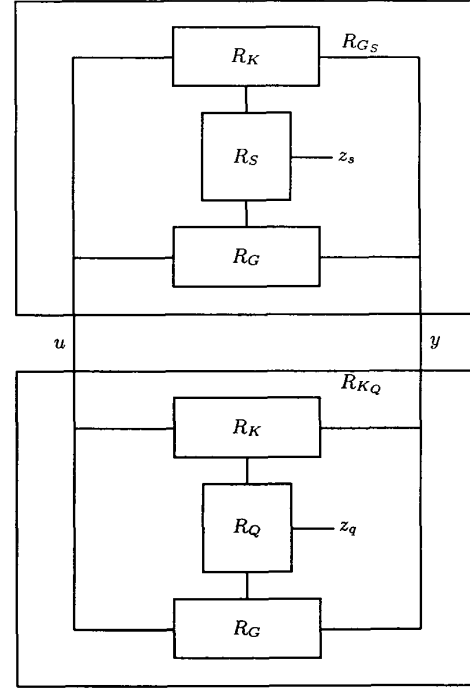


Figure 3: The Kernel Representation of $\{G_S, K_Q\}$.

Class of Stabilizing Plant-Controller Pairs

Consider the systems S and Q with skrs as follows

$$R_S : Z_K \times Z_G \rightarrow Z_S, \quad (8)$$

$$R_Q : Z_G \times Z_K \rightarrow Z_Q, \quad (9)$$

and initial condition spaces $\mathcal{X}_S, \mathcal{X}_Q$ respectively.

The systems G_S and K_Q are defined via their stable kernel representations R_{G_S} and R_{K_Q}

$$R_{G_S} : \mathcal{Y} \times \mathcal{U} \rightarrow Z_S \\ R_{G_S}(y, u) = R_S(R_G(y, u), R_K(u, y)), \quad (10)$$

$$R_{K_Q} : \mathcal{U} \times \mathcal{Y} \rightarrow Z_Q \\ R_{K_Q}(u, y) = R_Q(R_K(u, y), R_G(y, u)). \quad (11)$$

Note that the initial condition spaces of G_S and K_Q are $\mathcal{X}_S \times \mathcal{X}_{GK}$ and $\mathcal{X}_Q \times \mathcal{X}_{GK}$, respectively.

The properties of the feedback loop $\{G_S, K_Q\}$, as shown in Figure 3, are now investigated. This investigation yields the main results of the paper, and follow in a straightforward fashion from the definitions of the previous section.

Due to limitations of space, the proofs of the following theorems have not been included. The main ideas of the proof arise from considering the identity

$$R_{\{G_S, K_Q\}}^{x_{GK}SQ} = R_{\{S, Q\}}^{x_{SQ}} R_{\{G, K\}}^{x_{GK}}. \quad (12)$$

Theorem 3.1

Consider a well-posed system $\{G, K\}$ with skr (4), and systems S, Q , with skrs (8), (9) respectively, giving G_S

and K_Q with skrs (10), (11) respectively. Then the closed-loop system $\{G_S, K_Q\}$ is well-posed iff the closed loop system $\{S, Q\}$ is well-posed.

Further, given a well-posed system $\{G^*, K^*\}$ with skr $R_{\{G^*, K^*\}}$, there exist kernel representations R_{S^*} , R_{Q^*} , for the systems S^* and Q^* , given by

$$R_{S^*}^{z^*} = R_{G^*}^{z^*} \left[R_{\{G^*, K^*\}}^{z^*} \right]^{-1} : Z_{G^*} \rightarrow Z_{G^*}, \quad (13)$$

$$R_{Q^*}^{z^*} = R_{K^*}^{z^*} \left[R_{\{G^*, K^*\}}^{z^*} \right]^{-1} : Z_{G^*} \rightarrow Z_{K^*}. \quad (14)$$

such that $G_{S^*} = G^*$ and $K_{Q^*} = K^*$, and the system $\{S^*, Q^*\}$ is well-posed. If $\{G, K\}$ is generally stable, then R_{S^*} and R_{Q^*} will be stable kernel representations. \square

The previous theorem is now extended to give the stability properties of the closed loop $\{G_S, K_Q\}$.

Theorem 3.2

Consider a system $\{G, K\}$, with skr (4), which is well-posed and generally stable, and systems S, Q , with skrs (8), (9) respectively, giving G_S and K_Q with skrs (10), (11) respectively. Then the closed-loop system $\{G_S, K_Q\}$ will be well-posed and generally stable iff the closed-loop system $\{S, Q\}$ is well-posed and generally stable.

Further, given a system $\{G^*, K^*\}$, with skr $R_{\{G^*, K^*\}}$: $(u, y) \mapsto z_{G^*, K^*}$, which is well-posed and generally stable there exist stable kernel representations for the systems S^* and Q^* given by (13), (14) resp., such that $G_{S^*} = G^*$ and $K_{Q^*} = K^*$, and the system $\{S^*, Q^*\}$ is well-posed and generally stable. \square

Remark 3.3 Given a closed loop system $\{G, K\}$ with a stable kernel representation which is well-posed and generally stable, it is possible to parameterize the class of all well-posed and stable systems which have skrs. Thus these theorems give a generalization of the linear results of Tay, Moore and Horowitz [14], and the nonlinear results of Paice and Moore [6], when these are restricted to the noise free case.

Remark 3.4 That skrs lead to a parameterization of stable closed loop systems suggests a link to the theory of coprime factorizations. This is explored in [10]

Kernel Representations and Input-Output Operators

In this subsection the definitions required to specialize the framework presented in Section 2 to an input-output framework are presented. It is seen that the key to these results is to apply the definitions of well-posedness and stability for a closed loop to the system when in closed loop with the zero operator.

As noted previously, since we have not distinguished between the input and output spaces, the previous results may be considered from a behavioral point of view. In the case that we wish to move to an input-output or state space point of view, it becomes necessary to assume that it is possible to identify inputs and outputs, and that once the inputs are specified, the outputs are determined. This is equivalent to assuming that given a set of initial conditions $x \in \mathcal{X}_\Sigma$, each $z \in \mathcal{Z}$, yields an input-output map

$$\Sigma_z(x) : \mathcal{U} \rightarrow \mathcal{Y} \quad (15)$$

such that $y = \Sigma_z(x)u$ satisfies (3) for all $u \in \mathcal{U}$. This property is denoted *well-definedness* of the skr.

Definition 3.5 A kernel representation (1) is said to be well-defined if for each $z \in \mathcal{Z}$, and initial conditions $x \in \mathcal{X}_\Sigma$, (15) exists, so that for all $u \in \mathcal{U}$, $y = \Sigma_z(x)u$ iff $R_\Sigma^z(u, y) = z$. \square

Note that R_Σ can be well-defined for $x \in \mathcal{X}_\Sigma$ only if the map

$$R_\Sigma^z(\cdot, u) : \mathcal{Y} \rightarrow \mathcal{Z} \quad (16)$$

is one to one, and onto, i.e. invertible. We denote this inverse

$$[R_\Sigma^z]^{-1}(\cdot, u) : \mathcal{Z} \rightarrow \mathcal{Y}. \quad (17)$$

This is summarized in the following result:

Proposition 3.6 A given kernel representation (1) of Σ is well-defined iff for all $x \in \mathcal{X}$ and all $u \in \mathcal{U}$, the map $[R_\Sigma^z]^{-1}(\cdot, u)$ of (17) exists. \square

The proof is trivial and is left to the reader.

We will also need to discuss the stability of an input-output operator. This is defined as follows:

Definition 3.7 A system Σ with stable kernel representation $R_\Sigma(\cdot, \cdot)$, as in (1), is stable over the set $B \subset \mathcal{Z}^s \times \mathcal{X}_\Sigma$ if for all $(z, x) \in \mathcal{Z}^s \times \mathcal{X}_\Sigma$ the input-output map $\Sigma_z(x)$ is stable iff $(z, x) \in B$. \square

The system Σ with skr (1) is called *generally stable*, or simply *stable*, if it is stable over $\mathcal{Z}^s \times \mathcal{X}$.

By considering the zero operator in closed loop with another system it is possible to relate well-definedness with well-posedness, and system stability with the previous definition of closed-loop stability. This property is presented in Lemma 3.8. We first define a well-defined skr for the zero operator, 0, defined by

$$0 : \mathcal{U} \rightarrow \mathcal{Y}, \forall u \in \mathcal{U}, \quad 0(u) = 0. \quad (18)$$

as being given by

$$R_0(y, u) = y. \quad (19)$$

Lemma 3.8 Consider a system Σ with skr $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_\Sigma$, which is placed in closed loop with the system $0 : \mathcal{Y} \rightarrow \mathcal{U}$, with skr $R_0(u, y) = u$ (note that this is the reverse case to (19)). Then

1. R_Σ is well-defined iff the closed loop $\{\Sigma, 0\}$ is well-posed
2. The operator $\Sigma_z : \mathcal{U} \rightarrow \mathcal{Y}$ is stable over $B_\Sigma \subset \mathcal{Z}_\Sigma^s \times \mathcal{X}_\Sigma$ iff the feedback system $\{\Sigma, 0\}$ is stable over $B_\Sigma \times \mathcal{Z}_0^s$. \square

Note that as the zero operator has no state space, stating that $\{\Sigma, 0\}$ is stable over $B_\Sigma \times \mathcal{Z}_0^s$ is consistent with Definition 2.7.

The corollaries to Theorem 3.2 derived by considering alternately $S = 0$ and $Q = 0$ are now expressible in a form more easily seen to be generalizations of the existing results giving the Youla parameterization.

The Youla Parameterization via Stable Kernel Representations

The following corollaries to Theorem 3.2, give the class of all controllers which stabilize a given plant, and the class

of all plants stabilized by a given controller, respectively. They are generalizations of the results given in [6, 7], which were the first results giving Youla parameterizations for general nonlinear systems.

Corollary 3.9 Consider a system $\{G, K\}$ with skr (4), well-posed and generally stable, and the system Q , with skr (9) such that K_Q is given by the skr (11). Then the closed-loop system $\{G, K_Q\}$ will be well-posed iff the skr for Q is well-defined, and $\{G, K_Q\}$ will be generally stable iff Q is well-defined and stable.

Further, given a K^* , with skr $R_{K^*} : (u, y) \mapsto z_{K^*}$, the closed loop system $\{G, K^*\}$ is well-posed iff the kernel representation for the system Q^* given by (14) is well-defined. If the system $\{G, K^*\}$ is generally stable, then Q^* is stable. \square

Corollary 3.10 Consider a system $\{G, K\}$ with skr (4), which is well-posed and generally stable, and the system S , with skr (8) such that G_S is given by the skr (10). Then the closed-loop system $\{G_S, K\}$ will be well-posed iff the system S is well-defined, and $\{G_S, K\}$ will be generally stable iff S is well-defined and stable.

Further, given a G^* , with skr $R_{G^*} : (u, y) \mapsto z_{G^*}$, the closed loop system $\{G^*, K\}$ is well-posed iff the kernel representation for the system S^* given by (13) is well-defined. If the system $\{G^*, K\}$ is generally stable, then S^* is stable. \square

Remark 3.11 These corollaries give generalizations of the results presented in [6] to the stable kernel representation framework. They give explicit versions of the Youla parameterization for linear systems. Further, as seen in section 4, it is possible to derive skrs for nonlinear systems with general state space representations. By applying these corollaries to this special case, a state space characterization of the Youla parameterization for nonlinear systems may be derived. We believe that these are the first such results presented in the literature.

4. State Space Results

In this section we present some state space results which were recently obtained by Scherpen and van der Schaft [11]. A skr is derived for a general nonlinear system with a state space description. This overcomes a major weakness of the nonlinear left factorization theory, where a method for deriving left factorizations from a state space realization of a nonlinear operator has not been derived.

Consider a nonlinear system $G : \mathcal{U} \rightarrow \mathcal{Y}$ which has state space description:

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \quad (20)$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and $x = (x_1, \dots, x_n)$ are local coordinates for a smooth state space manifold, M . G defines an input-output map $G(x_0)$ when the initial condition $x(0) = x_0$ is specified. It is assumed that the system has an equilibrium, without loss of generality this taken is to be zero, i.e. $f(0) = 0$, and that $h(0) = 0$.

The equation $z = h(x) - y$ is considered in order to derive a stable kernel representation. This is motivated by the linear theory, where transforming the state equations such that the map $(u, y) \mapsto z$ is input to state stable, and $z = 0$

for $y = Gu$, yields a stable left factorization of the original system, see [5].

In [11], what is here denoted a *stable kernel representation* is denoted a *left coprime factorization*, we shall retain the notation of the previous sections. Additionally, a special form, a *normalized left coprime factorization* is dealt with in [11]. In order to define this, it is necessary to define the notion of a *co-inner* nonlinear system. A detailed consideration of these conditions is beyond the scope of this paper, so we work with the following definition of coprimeness.

Definition 4.1 A coprime kernel representation of a nonlinear system (20) is a system of the form

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x}) \begin{bmatrix} u \\ y \end{bmatrix} \\ z &= \tilde{h}(\tilde{x}) - y \end{aligned} \quad (21)$$

where \tilde{f} is Lyapunov stable, the input-output map for every initial condition is L_2 -stable, the dynamics resulting from the constraint $z = h(\tilde{x}) - y = 0$, i.e.

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}_1(\tilde{x}) + \tilde{g}_2(\tilde{x})\tilde{h}(\tilde{x}) \\ y &= \tilde{h}(\tilde{x}) \end{aligned} \quad (22)$$

equals (20), and there exists a right-inverse for (21)

$$\begin{aligned} \tilde{p} &= \tilde{f}(\tilde{p}) + \tilde{g}(\tilde{p})s \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} \tilde{h}_1(\tilde{p}) \\ \tilde{h}(\tilde{p}) - s \end{bmatrix}, \end{aligned} \quad (23)$$

with \tilde{f} Lyapunov stable. \square

The following two Hamilton-Jacobi-Bellman equations are introduced in relation to the system (20).

$$\frac{\partial V}{\partial x}(x)f(x) - \frac{1}{2} \frac{\partial V}{\partial x}(x)g(x)g(x)^T \frac{\partial^T V}{\partial x}(x) + \frac{1}{2} h(x)^T h(x) = 0 \quad (24)$$

$$\frac{\partial W}{\partial x}(x)f(x) + \frac{1}{2} \frac{\partial W}{\partial x}(x)g(x)g(x)^T \frac{\partial^T W}{\partial x}(x) - \frac{1}{2} h(x)^T h(x) = 0 \quad (25)$$

$V(0) = 0$, $W(0) = 0$. It is assumed that (24) and (25) have smooth non-negative definite solutions, W and V resp., at least on a neighborhood of 0 (see also Remark 4.4). Based on these solutions a coprime kernel representation may be derived for the system (20).

Theorem 4.2

Let V and W be smooth positive definite solutions (i.e. $V(x) > 0$, $W(x) > 0$, $x \neq 0$) to the Hamilton-Jacobi-Bellman equations (24), (25), respectively. Since $\frac{\partial W}{\partial x}(0) = 0$ and $h(0) = 0$, there exist smooth matrices $M(x)$ and $C(x)$, such that

$$\frac{\partial W}{\partial x}(x) = x^T M(x), \quad h(x) = C(x)x. \quad (26)$$

Assume that $M(x)$ is invertible for all x . Then a coprime kernel representation of the system (20) is given by

$$\begin{aligned} \dot{\tilde{x}} &= f(x) - M^{-1}(x)C^T(x)h(x) + g(x)u + M^{-1}(x)C^T(x)y \\ z &= h(x) - y, \end{aligned} \quad (27)$$

where $f(x) - M^{-1}(x)C(x)^T h(x)$ is Lyapunov stable with Lyapunov function W . Furthermore, an internally stable

right inverse of (27) is given by

$$\begin{aligned} \dot{p} &= f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p) - M^{-1}(p)C^T(p)s \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} -g^T(p)\frac{\partial^T V}{\partial p}(p) \\ h(p) - s \end{bmatrix} \end{aligned} \quad (28)$$

□

Remark 4.3 Furthermore, if V and W are proper (i.e. for each $c > 0$ the set $\{x \in M \mid 0 \leq V(x) \leq c\}$ is compact, and similarly for W), and $\dot{x} = f(x)$, $y = h(x)$ is zero-state detectable (i.e. $y(t) \equiv 0$, implies that $\lim_{t \rightarrow \infty} x(t) = 0$), then $f(x) - M^{-1}(x)C^T(x)h(x)$ and $f(p) - g(p)g^T(p)\frac{\partial^T V}{\partial p}(p)$ are globally asymptotically stable.

Remark 4.4 If the linearization of (20) at $x = 0$ is controllable and observable, then at least on a neighborhood of $x = 0$ there exist smooth positive definite solutions V , W to (24), (25) respectively.

Remark 4.5 In the case of a linear system (20), the left coprime factorization (27) reduces to the state space representation of a normalized left coprime factorization, see Vidyasagar [16].

Thus, at least in a local setting, there exists a procedure for deriving a stable kernel representation for a general nonlinear system. This may be applied to the results of the previous sections, giving state space version of the Youla parameterization for nonlinear systems.

5. Conclusions

In this paper we have developed the theory of stable kernel representations for nonlinear systems, and demonstrated that they are a generalization of left coprime factorizations for linear systems. The results presented in Section 3. demonstrate that in the noise-free case it is possible to duplicate the main results of the nonlinear left factorization theory simply by replacing the left factorizations by stable kernel representations. Specifically, the Youla parameterization of all stabilizing plant-controller pairs has been shown to result from this approach.

As further support for this approach to Nonlinear Control, a derivation of a stable kernel representation for a general nonlinear plant is presented from Scherpen and van der Schaft [11]. It is a simple exercise to see that this leads to state space representations for the Youla parameterization for nonlinear systems.

It is expected that results in Nonlinear Robust Control may be derived from the results presented in this paper, as was the case with the results for nonlinear left coprime factorizations, and that the many useful techniques which result in the linear theory due to the use of coprime factorization analysis may now be derived in a nonlinear form.

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